62-4-6

RESEARCH REPORT 19
1 MAY 1962
I.E.R. 172-22

SYMMETRIC DUAL QUADRATIC PROGRAMS

by

Richard W. Cottle

OPERATIONS RESEARCH CENTER

INSTITUTE OF ENGINEERING RESEARCH

UNIVERSITY OF CALIFORNIA - BERKELEY

SYMMETRIC DUAL QUADRATIC PROGRAMS

by

Richard W. Cottle
Operations Research Center
University of California, Berkeley

1 May 1962 Research Report 19

This research has been partially supported by the Office of Naval Research under Contract Nonr-222(83) with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

SYMMETRIC DUAL QUADRATIC PROGRAMS

I. Introduction

The duality theory of quadratic programming has been studied by Dennis [1] and principally by Dorn [2,3,4]. In [7], Wolfe treated the restriction of his results in nonlinear duality to the case of quadratic programming.

In this paper, two quadratic programs are presented which are dual, naturally symmetric, and related to a self-dual quadratic program. It is a consequence of the duality of these programs that if either has an optimal solution, then they share an optimal solution in common. Since duality, symmetry, and self-duality have each been studied by Dorn, some attention is given to the relation between the present work and his.

Consider the quadratic programs:

PRIMAL PROBLEM

(1a) Minimize $F(x,y) = \frac{1}{2}\, x'Cx + \frac{1}{2}\, y'C^{*}y + p'x$

(1b) subject to $Ax + C^{*}y \geq -b$

(1c) and $x \geq 0$ (y unrestricted) ;

DUAL PROBLEM

(2a) Maximize $G(u,v) = -\frac{1}{2}\, u'Cu - \frac{1}{2}\, v'C^{*}v - b'v$

(2b) subject to $Cu - A'v \geq -p$

(2c) and $v \geq 0$ (u unrestricted) ;

COMBINED SELF-DUAL PROBLEM

(3a) Minimize $H(x,y) = y'C^{*}y + x'Cx + p'x + b'y$

(3b) subject to $C^{*}y + Ax \geq -b$,

(3c) $-A'y + Cx \geq -p$,

(3d) $y \geq 0$,

(3e) and $x \geq 0$.

Notation and Terminology

The entries of all matrices and vectors are real numbers. The matrices A, C, and C^* are $m \times n$, $n \times n$, and $m \times m$, respectively. The latter two are assumed to be symmetric with positive semi-definite quadratic forms. The symbols x, u, or p and y, v, or b denote n- and m-component column vectors, respectively. The components of x, y, u, and v are variables, whereas those of A, C, C^*, p, and b are constants. The transpose of a matrix or a vector is denoted by a prime. An inequality between vectors means that the stated inequality holds between each of the corresponding components. The symbol Φ denotes the empty set.

Problem (1) is called the primal problem, and (2) is called the dual problem. A pair (x,y) of vectors is called feasible for (1) if it satisfies (1b) and (1c). Similarly, a pair (u,v) is feasible for (2) if it satisfies (2b) and (2c). The set of all feasible pairs of vectors for the primal problem (1) is called its constraint set, $\mathcal{P}$. The constraint set of the dual (2) is denoted by $\mathcal{D}$. If the constraint set of a problem is empty (i.e., $\mathcal{P} = \Phi$ or $\mathcal{D} = \Phi$) the problem is said to be infeasible. A problem is solvable if its constraint set contains a pair for which the objective function (F in (1) or G in (2)) attains the desired extremum; such a pair is an optimal solution of the problem. The sets $\mathcal{P}_o \subset \mathcal{P}$ and $\mathcal{D}_o \subset \mathcal{D}$ denote the optimal solutions of (1) and (2), respectively.

It will be shown that a relation of duality holds between (1) and (2) in the sense (see [7]) that:

(i) $\sup G(u,v) \leq \inf F(x,y)$;

(ii) the solvability of one problem implies that of the other, and the extremal values of F and G are equal;

(iii) if one problem is feasible while the other is not, then on its constraint set, the objective function of the feasible problem is unbounded in the direction of extremization.

It is, of course, possible for both the primal and the dual problems to be infeasible (i.e., $\mathcal{P} = \mathcal{D} = \Phi$).

When C^* is the zero matrix, the dual problems of Dorn [2] result. When both C and C^* are zero matrices, the familiar von Neumann symmetric dual linear programs appear.

Problem (2) can be converted to a minimization problem:

$$
\begin{array}{llrcll}
\text{(4a)} & \text{Minimize} & -G(u,v) = \frac{1}{2} v'C^*v + \frac{1}{2} u'Cu + b'v & & & \\
\text{(4b)} & \text{subject to} & (-A')v + Cu & \geq & -p & \\
\text{(4c)} & \text{and} & v & \geq & 0 & (u \text{ unrestricted}) .
\end{array}
$$

With (4) as the primal problem, the dual is:

$$
\begin{array}{llrcll}
\text{(5a)} & \text{Maximize} & -F(x,y) = -\frac{1}{2} y'C^*y - \frac{1}{2} x'Cx - p'x & & & \\
\text{(5b)} & \text{subject to} & C^*y - (-A')'x & \geq & -b & \\
\text{(5c)} & \text{and} & x & \geq & 0 & (y \text{ unrestricted}) .
\end{array}
$$

But (5) is obviously equivalent to (1). Roughly speaking, then, the dual of the dual is the primal. It is precisely this involutory property which Dorn [3] calls symmetry.

II. Duality

THEOREM 1: $\sup\ G(u,v) \leq \inf\ F(x,y)$.

PROOF: Using the convention of [7] that

$$
\begin{array}{lll}
\text{if } \mathcal{D} = \Phi & , & \sup\ G = -\infty \\
\text{if } \mathcal{P} = \Phi & , & \inf\ F = +\infty
\end{array}
$$

it remains to prove the inequality under the assumption that both problems are feasible. Let $(x,y) \in \mathcal{P}$ and $(u,v) \in \mathcal{D}$. Since $x \geq 0$ and $v \geq 0$,

$$-b'v - y'C^*v \leq x'A'v \leq p'x + x'Cu \quad .$$

Applying the inequalities

$$2y'C^*v \leq y'C^*y + v'C^*v$$
$$2x'Cu \leq x'Cx + u'Cu$$

which follow from $(y - v)'C^*(y - v) \geq 0$ and $(x - u)'C(x - u) \geq 0$ because C^* and C are symmetric and positive semi-definite,

$$-b'v - \tfrac{1}{2} y'C^*y - \tfrac{1}{2} v'C^*v \leq p'x + \tfrac{1}{2} x'Cx + \tfrac{1}{2} u'Cu$$

and by transposing,

$$G(u,v) = -\tfrac{1}{2} u'Cu - \tfrac{1}{2} v'C^*v - b'v \leq \tfrac{1}{2} x'Cx + \tfrac{1}{2} y'C^*y + p'x = F(x,y) \quad .$$

This completes the proof of Theorem 1.

A solvable quadratic program is related to a certain <u>linear</u> program. By means of this correspondence, the duality theorem of linear programming may be employed.

LEMMA: Let (x_o,y_o) be an optimal solution of (1). Then (x_o,y_o) is an optimal solution of the linear program:

(6a)	Minimize	$f(x,y) = (x_o'C)x + (y_o'C^*)y + p'x$
(6b)	subject to	$Ax + C^*y \geq -b$
(6c)	and	$x \geq 0$.

REMARK: This Lemma is a special case of one where the objective function F in (1a) is replaced by a differentiable convex function and the linear objective form in (6a) is replaced by the gradient of the new objective function (transposed) times the variables. The proofs of the two Lemmas are completely analogous; that for the quadratic function F will be given here.

PROOF: (1) and (6) have the same constraint set, $\mathcal{P}$. Suppose these exists $(\bar{x},\bar{y}) \in \mathcal{P}$ such that $f(\bar{x},\bar{y}) - f(x_o,y_o) < 0$. This means

$$(7) \qquad (x_o'C + p')(\bar{x} - x_o) + (y_o'C^*)(\bar{y} - y_o) < 0 \quad .$$

Let $0 < \lambda < 1$, and define

$$x^* = (1 - \lambda)x_o + \lambda\bar{x} = x_o + \lambda(\bar{x} - x_o) \quad ,$$

$$y^* = (1 - \lambda)y_o + \lambda\bar{y} = y_o + \lambda(\bar{y} - y_o) \quad .$$

Since $\mathcal{P}$ is convex, $(x^*,y^*) \in \mathcal{P}$. Consider

$$(8) \qquad F(x^*,y^*) - F(x_o,y_o) = \lambda[(x_o'C + p')(\bar{x} - x_o) + (y_o'C^*)(\bar{y} - y_o)] + \lambda^2[(\bar{x} - x_o)'C(\bar{x} - x_o) + (\bar{y} - y_o)'C^*(\bar{y} - y_o)] \quad .$$

From (7) it follows that the right-hand side of (8) can be made negative for sufficiently small, positive λ . This contradicts the assumption that (x_o,y_o) is an optimal solution of (1). Therefore, (x_o,y_o) must be optimal for (6).

THEOREM 2: <u>If</u> (x_o,y_o) <u>is an optimal solution of (1), there exists an optimal solution of (2). Furthermore, the extremal values of</u> F <u>and</u> G <u>are equal</u>.

PROOF: By Theorem 1, if there exists $(u,v) \in \mathcal{D}$ such that $G(u,v) = F(x_o,y_o)$, then $(u,v) \in \mathcal{D}_o$. By the Lemma, (x_o,y_o) is optimal for (6). The duality theorem of linear programming (cf. e.g., [6]) states that there exists a vector v_o such that

$$Cx_o - A'v_o \geq -p$$

$$v_o \geq 0$$

$$v_o'C^* = y_o'C^*$$

and

$$(9) \qquad -b'v_o = x_o'Cx_o + y_o'C^*y_o + p'x_o \quad .$$

Notice that $(x_o,v_o) \in \mathcal{D}$. Also, by the symmetry of C^* ,

$$v_o'C^*v_o = v_o'C^*y_o = y_o'C^*v_o = y_o'C^*y_o \quad .$$

Thus, from (9)

$$\begin{aligned} F(x_o,y_o) &= \tfrac{1}{2}\, x_o'Cx_o + \tfrac{1}{2}\, y_o'C^*y_o + p'x_o \\ &= -\tfrac{1}{2}\, x_o'Cx_o - \tfrac{1}{2}\, y_o'C^*y_o - b'v_o \\ &= -\tfrac{1}{2}\, x_o'Cx_o - \tfrac{1}{2}\, v_o'C^*v_o - b'v_o \\ &= G(x_o,v_o) \quad , \end{aligned}$$

and the proof is complete.

The demonstration of Theorem 2 provides the important

COROLLARY 1: If $(x_o,y_o) \in \mathcal{P}_o$, there exists a vector v_o such that $(x_o,v_o) \in \mathcal{D}_o$, and $G(x_o,v_o) = F(x_o,y_o)$.

This Corollary and the symmetry established above yield

COROLLARY 2: If $(u_o,v_o) \in \mathcal{D}_o$, there exists a vector x_o such that $(x_o,v_o) \in \mathcal{P}_o$, and $F(x_o,v_o) = G(u_o,v_o)$.

COROLLARY 3: Nonnegativity restrictions may be extended to the variables y or u without affecting the question of solvability of (1) or (2).

THEOREM 3: <u>If (1) is feasible and (2) is infeasible, then</u> $\inf_{(x,y)\in\mathcal{P}} F(x,y) = -\infty$

PROOF: Let $(\bar{x},\bar{y}) \in \mathcal{P}$. The assumption that $\mathcal{D} = \Phi$ means that there exist no vectors u and v satisfying

$$\begin{pmatrix} C & -A' \\ 0 & I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \geq \begin{pmatrix} -p \\ 0 \end{pmatrix} \quad .$$

It follows (cf. [6], p. 46) that there exist vectors $x^* \geq 0$ and $y^* \geq 0$ such that

$$\begin{pmatrix} C & 0 \\ -A & I \end{pmatrix} \begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad ,$$

$$(-p',0) \begin{pmatrix} x^* \\ y^* \end{pmatrix} = 1 \quad .$$

In particular,

$$\begin{aligned} Iy^* = Ax^* &\geq 0 \\ Cx^* &= 0 \\ p'x^* &= -1 \quad . \end{aligned}$$

With $\lambda \geq 0$, it follows that $(\bar{x} + \lambda x^* , \bar{y}) \in \mathcal{P}$. However,

$$F(\bar{x} + \lambda x^* , \bar{y}) = -\lambda + F(\bar{x},\bar{y}) \quad .$$

Clearly, $\lim_{\lambda \to \infty} F(\bar{x} + \lambda x^* , \bar{y}) = -\infty$.

Again the symmetric result holds.

COROLLARY 1: If (2) is feasible and (1) is infeasible, then $\sup_{(u,v) \in \mathcal{D}} G(u,v) = \infty$.

Theorems 1 and 3 and Corollary 1 have the immediate consequence

COROLLARY 2: If either problem (1) or (2) is feasible, its objective function is bounded in the direction of extremization if and only if the other problem is feasible.

The duality relation is now fully established.

In view of Theorem 2 and its Corollaries it is not surprising to find that if either (1) or (2) is solvable, there exists a common optimal solution for (1) and (2).

THEOREM 4 (Joint Solution): <u>If either problem (1) or problem (2) is solvable, there exist vectors x_o and v_o such that (x_o, v_o) is optimal for (1) and (2).</u> (In this case, (x_o, v_o) is called a joint solution.)

PROOF: It is no restriction to assume that $\mathcal{P}_o$ is nonempty; let $(x_o, y_o) \in \mathcal{P}_o$. Then there exists a vector v_o such that $(x_o, v_o) \in \mathcal{D}_o$. Hence,

$$(10) \qquad \begin{aligned} Ax_o + C^* y_o &\geq -b \quad , \quad x_o \geq 0 \\ Cx_o - A'v_o &\geq -p \quad , \quad v_o \geq 0 \end{aligned}$$

and

$$\tfrac{1}{2}\, x_o'Cx_o + \tfrac{1}{2}\, y_o'C^* y_o + p'x_o = -\tfrac{1}{2}\, x_o'Cx_o - \tfrac{1}{2}\, v_o'C^* v_o - b'v_o \quad .$$

The latter implies

$$(11) \qquad x_o'Cx_o + p'x_o = -\tfrac{1}{2}\, y_o'C^* y_o - \tfrac{1}{2}\, v_o'C^* v_o - b'v_o \quad ,$$

and the inequalities (10) imply

$$x_o'Cx_o + p'x_o \geq v_o'Ax_o \geq -v_o'C^* y_o - b'v_o \geq -\tfrac{1}{2}\, y_o'C^* y_o - \tfrac{1}{2}\, v_o'C^* v_o - b'v_o \quad .$$

By (11), equality holds throughout and

$$v_o'C^* y_o = \tfrac{1}{2}\, y_o'C^* y_o + \tfrac{1}{2}\, v_o'C^* v_o \quad .$$

Therefore $(v_o - y_o)'C^*(v_o - y_o) = 0$, from which $C^* v_o = C^* y_o$ follows. (See App. g, [5].) Hence $(x_o, v_o) \in \mathcal{P}$, and $F(x_o, v_o) = F(x_o, y_o)$, so $(x_o, v_o) \in \mathcal{P}_o$.

REMARK (Complementary Slackness): Let (x_o, v_o) be a joint solution of (1) and (2). Define

$$s_o = Ax_o + C^* v_o + b \quad ,$$

$$t_o = Cx_o - A'v_o + p \quad .$$

Then

(12) $$s_o' v_o = 0 \; , \quad \text{and} \quad t_o' x_o = 0 \quad .$$

This property, can be deduced directly or as a consequence of the Kuhn-Tucker conditions.

In the case of linear programming, the feasibility of both the primal and the dual problems implies the existence of optimal solutions for each of them. Although this is not generally true in nonlinear programming, it <u>is</u> in quadratic programming.

THEOREM 5: If both (1) and (2) are feasible, then the infinimum of F over $\mathcal{P}$ is attained on $\mathcal{P}$ and the supremum of G over $\mathcal{D}$ is attained on $\mathcal{D}$. (Moreover, (1) and (2) have a joint solution.)

PROOF: Because $\mathcal{P}$ and $\mathcal{D}$ are each nonempty, F is bounded below on $\mathcal{P}$, and G is bounded above on $\mathcal{D}$. To the latter, with some variations in the proof, may be applied App. i of [5], to the effect that G must attain its supremum (over $\mathcal{D}$) on $\mathcal{D}$. The remainder of the proof is an application of Corollary 2, Theorem 2. The assertion in parentheses is true by Theorem 4.

III. Self-Duality

Consider a (primal) problem of the following kind:

(13a) Minimize $\varphi(z) = z'Bz + q'z$

(13b) subject to $Bz \geq -q$

(13c) and $z \geq 0$

where B is positive semi-definite, but <u>not necessarily symmetric</u>. If B were symmetric, problem (13) would be a special case of (1). The symmetry can, indeed, be brought about with no adverse effects on the problem. Set

$$\bar{B} = \tfrac{1}{2}(B + B') \quad .$$

It is easy to verify that:

(i) $\bar{B}$ is symmetric;
(ii) $\bar{B}$ is positive semi-definite;
(iii) $x'\bar{B}x = x'Bx$, for all x .

Therefore (13) is equivalent to:

(14a) Minimize $\varphi(z) = z'\bar{B}z + q'z$

(14b) subject to $Bz \geq -q$

(14c) and $z \geq 0$.

Dorn [4] has proved that a problem of the type (13), in which B is further assumed to be positive definite, is always solvable, has $\text{Min } \varphi(z) = 0$, and is self-dual. If the hypothesis of positive definiteness is exchanged for positive semi-definiteness <u>and</u> feasibility, the same conclusions are valid.

THEOREM 6: <u>If (14) is feasible it is solvable and</u> $\text{Min } \varphi(z) = 0$.

PROOF: For any feasible z , (14b) and (14c) imply $\varphi(z) \geq 0$; since φ is bounded below (14) has an optimal solution. The dual of (14) is

(15a) Maximize $\psi(u,v) = -u'\bar{B}u - q'v$

(15b) subject to $2\bar{B}u - B'v \geq -q$

(15c) and $v \geq 0$,

and it is solvable. Multiplying (15b) by v and subtracting $u'\bar{B}u$ from both sides of the inequality yield

$$-u'\bar{B}u + 2v'\bar{B}u - v'B'v \geq -u'\bar{B}u - q'v$$

from which it follows that

$$(16) \qquad 0 \geq -(u-v)'\bar{B}(u-v) \geq -u'\bar{B}u - q'v = \psi(u,v) \quad .$$

Now

$$0 \leq \text{Min } \varphi(z) = \text{Max } \psi(u,v) \leq 0$$

so that Min $\varphi(z) = 0$.

COROLLARY: If (14) is infeasible, so is (15).

PROOF: In the proof of the Theorem it was shown that $\psi(u,v) \leq 0$ whenever (u,v) satisfy ~~(15b)~~ and (15c). But ψ cannot be bounded above if (14) is infeasible.

This Corollary shows that (14) and (15) are either both feasible or both infeasible. Hence it is sufficient to consider only the case where (14) and (15) are both feasible — and hence solvable. Let $\mathcal{P}_o$ and $\mathcal{D}_o$ denote the sets of optimal solutions of (14) and (15), respectively.

THEOREM 7: Problem (14) is self-dual in the sense that

(i) $z_o \in \mathcal{P}_o$ implies $(z_o, z_o) \in \mathcal{D}_o$,

(ii) $(u_o, v_o) \in \mathcal{D}_o$ implies $v_o \in \mathcal{P}_o$.

PROOF:

(i) If $z_o \in \mathcal{P}_o$, $\varphi(z_o) = 0$. Now

$$2\bar{B}z_o - B'z_o = Bz_o \geq -q \quad , \quad \text{and} \quad z_o \geq 0$$

so $(z_o, z_o) \in \mathcal{D}$. Moreover,

$$\psi(z_o, z_o) = -z_o'\bar{B}z_o - q'z_o = -\varphi(z_o) = 0 \quad ,$$

and therefore $(z_o, z_o) \in \mathcal{D}_o$.

(ii) Let $(u_o, v_o) \in \mathcal{D}_o$. Inequality (16) and the fact that $\psi(u_o, v_o) = 0$ imply

$$\bar{B}u_o = \bar{B}v_o \quad .$$

This and the symmetry of $\bar{B}$ imply

$$u_o'\bar{B}u_o = v_o'\bar{B}v_o \quad .$$

Now $v_o \geq 0$, and

$$Bv_o = (Bv_o + B'v_o) - B'v_o = 2\bar{B}v_o - B'v_o \geq -q \quad .$$

Finally,

$$\varphi(v_o) = v_o'\bar{B}v_o + q'v_o = -\psi(v_o, v_o) = 0 \quad ,$$

so that $v_o \in \mathcal{P}_o$.

The self-dual program (14) is not entirely artificial, for problem (3) stated in Section I is of the form (13). This can be seen by setting

$$B = \begin{pmatrix} C^* & A \\ -A' & C \end{pmatrix} \quad , \quad q = \begin{pmatrix} b \\ p \end{pmatrix} \quad , \quad z = \begin{pmatrix} y \\ x \end{pmatrix} \quad ,$$

and $\varphi(z) = H(x,y) = F(x,y) - G(x,y)$. Theorem 7 shows that (3), by modification to the form (14), is (equivalent to) a self-dual program.

The sense of self-duality defined in Theorem 7 is not thoroughly satisfactory. It is known that by combining the symmetric primal and dual problems of linear programming, it is possible to obtain a formally self-dual problem. Indeed, if $\bar{A}$ is a skew-symmetric matrix, the linear program

Moreover

$$\Psi(U,V) = -\Phi(U,V) \leq 0$$

where Ψ is the dual objective function, (19a). Now

$$0 \leq \mathrm{Min} \quad (X,Y) = \mathrm{Max} \quad (U,V) \leq 0 \quad ,$$

whence the conclusion.

A <u>true</u> combination of (1) and (2) is:

(21a) Minimize $K(x,y,u,v) = \frac{1}{2}y'C^{*}y + \frac{1}{2}u'Cu + \frac{1}{2}v'C^{*}v + \frac{1}{2}x'Cx + b'v + p'x$

(21b) subject to $C^{*}y + Ax \geq -b \quad ,$

(21c) $-A'v + Cu \geq -p \quad ,$

(21d) $v \geq 0 \quad ,$

(21e) and $x \geq 0 \quad .$

Under the substitutions,

$$\bar{A} = \begin{pmatrix} 0 & A \\ -A' & 0 \end{pmatrix} , \quad \bar{C} = \begin{pmatrix} C^{*} & 0 \\ 0 & C \end{pmatrix} , \quad q = \begin{pmatrix} b \\ p \end{pmatrix} , \quad X = \begin{pmatrix} v \\ x \end{pmatrix} , \quad Y = \begin{pmatrix} y \\ u \end{pmatrix}$$

it is clear that (21) is of the form (18) and is, therefore, self-dual.

To answer the question of the relation between the self-dual problems (14) and (18), note that the former can be written as

$$\begin{aligned} &\text{Minimize} && \varphi(z) = \tfrac{1}{2} z'\bar{B}z + \tfrac{1}{2} z'\bar{B}z + q'z \\ &\text{subject to} && \bar{B}z + (\bar{B} - B')z \geq -q \\ &\text{and} && z \geq 0 \quad . \end{aligned}$$

Now $\bar{B}$ is symmetric and positive semi-definite, and $(\bar{B} - B')$ is skew symmetric. Thus let $\bar{A} = (\bar{B} - B')$ and let $\bar{C} = \bar{B}$. The problem above becomes, then,

$$\text{(17a)} \quad \text{Minimize} \quad q'z$$

$$\text{(17b)} \quad \text{subject to} \quad \bar{A}z \geq -q$$

$$\text{(17c)} \quad \text{and} \quad z \geq 0$$

is clearly self-dual. This formal self-duality can be shown in quadratic programming as well. Consider the problem:

$$\text{(18a)} \quad \text{Minimize} \quad \Phi(X,Y) = \tfrac{1}{2} Y'\bar{C}Y + \tfrac{1}{2} X'\bar{C}X + q'X$$

$$\text{(18b)} \quad \text{subject to} \quad \bar{C}Y + \bar{A}X \geq -q$$

$$\text{(18c)} \quad \text{and} \quad X \geq 0$$

where $\bar{A}$ is skew symmetric and $\bar{C}$ is symmetric and positive semi-definite. The dual of (18) is:

$$\text{(19a)} \quad \text{Maximize} \quad \Psi(U,V) = -\tfrac{1}{2} V'\bar{C}V - \tfrac{1}{2} U'\bar{C}U - q'V$$

$$\text{(19b)} \quad \text{subject to} \quad -\bar{A}'V + \bar{C}U \geq -q$$

$$\text{(19c)} \quad \text{and} \quad V \geq 0 \quad .$$

Since $\bar{A}$ is skew symmetric, (19) is just (18); that is, (18) is self-dual.

THEOREM 8: <u>If (18) is feasible</u>, Min $\Phi(X,Y) = 0$.

PROOF: Since $\bar{A}$ is skew-symmetric, $X'\bar{A}X = 0$ for <u>all</u> X . Let (X,Y) be feasible for (18). Taking transposes in (18) and multiplying by X , it follows that

$$Y'\bar{C}X \geq -q'X \quad .$$

Therefore,

$$\text{(20)} \quad \Phi(X,Y) = \tfrac{1}{2} Y'\bar{C}Y + \tfrac{1}{2} X'\bar{C}X + q'X \geq (Y - X)'\bar{C}(Y - X) \geq 0 \quad .$$

Since Φ is bounded below for feasible vectors, (18) is solvable — and so is (19).

(22a)	Minimize	$\varphi(z) = \frac{1}{2} z'\bar{C}z + \frac{1}{2} z'\bar{C}z + q'z$
(22b)	subject to	$\bar{C}z + \bar{A}z \geq -q$
(22c)	and	$z \geq 0$.

Thus (14) and (22) are the same problem. If z is feasible for (14), (z,z) is feasible for (18). Furthermore, if z_o is optimal for (14), then (z_o,z_o) is optimal for (18) because $\varphi(z_o) = 0$, the minimum value of Φ .

On the other hand, if (X_o,Y_o) is an optimal pair for (18), then by Theorem 8 and Equation (20),

$$0 = \Phi(X_o,Y_o) \geq (Y_o - X_o)'\bar{C}(Y_o - X_o) \geq 0 \quad .$$

Therefore $\bar{C}X_o = \bar{C}Y_o$ (as seen in similar arguments above) so that

$$\varphi(X_o) = \Phi(X_o,X_o) = 0 \quad .$$

This means that X_o is feasible and optimal in (22), hence in (14).

These remarks show that (14) and (18) are equivalent problems. The proof of this equivalence closely parallels that of Theorem 7. It may be more accurate, therefore, to say that (14) is actually equivalent to its dual.

REFERENCES

[1] Dennis, Jack B., Mathematical Programming and Electrical Networks, Technology Press and John Wiley and Sons, New York, 1959.

[2] Dorn, W. S., "Duality in Quadratic Programming," Quarterly of Applied Mathematics, 18 (1960), pp. 155-162.

[3] Dorn, W. S., "A Symmetric Dual Theorem for Quadratic Programs," Journal of the Operations Research Society of Japan, 2 (1960), pp. 93-97.

[4] Dorn, W. S., "Self-Dual Quadratic Programs," Journal of the Society for Industrial and Applied Mathematics, 9 (1961), pp. 51-54.

[5] Frank, Marguerite, and Philip Wolfe, "An Algorithm for Quadratic Programming," Naval Research Logistics Quarterly, 3 (1956), pp. 95-110.

[6] Gale, David, The Theory of Linear Economic Models, McGraw-Hill, New York, 1960.

[7] Wolfe, Philip, "A Duality Theorem for Nonlinear Programming," Quarterly of Applied Mathematics, 19 (1961), pp. 239-244.

ADDITIONAL BIBLIOGRAPHY

1. Dorn, W. S., "A Duality Theorem for Convex Programs," IBM Journal of Research and Development, 4 (1960), pp. 407-413.

2. Eisenberg, E., "Duality in Homogeneous Programming," Proceedings of the American Mathematical Society, 12 (1961), pp. 783-787.

3. Eisenberg, E., "Closure of Certain Quasi-Linear Convex Bodies," Operations Research Center, University of California, Berkeley, RR 5, 20 December, 1961.

4. Hanson, M. A., "A Duality Theorem for Nonlinear Programming with Nonlinear Constraints," Australian Journal of Statistics, 3 (1961), pp. 64-72.

5. Huard, P., "Dual Programs," IBM Journal of Research and Development, 6 (1962), pp. 137-139.

6. Mangasarian, O. L., "Duality in Nonlinear Programming," Shell Development Company, Emeryville, California, Report P-1054.